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TRANSIENT RESPONSE AT THE BOUNDARY OF A CYLINDRICAL CAVITY IN AN ELASTIC MEDIUM

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Abstract—An unbounded, elastic medium with a circular, cylindrical cavity is subjected to a circumferentially uniform, suddenly applied step pressure pulse at the cavity surface. Formulae for the circumferential stress, radial displacement, and radial velocity at the cavity wall are developed. These formulae contain only elementary functions and can easily be used as influence coefficients to determine, by means of Duhamel integrals, the response produced by an arbitrary pressure pulse.

INTRODUCTION

IN ORDER to explain the first phase in the mechanism of rock blasting, Selberg [1] investigated the transient stress waves emanating from a cylindrical cavity in an elastic medium. A solution was obtained by using the Laplace-transform method and the inversion integral theorem. The evaluation of Selberg's solution is cumbersome because the zeros of an expression containing modified Bessel functions must be located and a line integral along the negative real axis must be calculated numerically. Selberg presents stress response data for an elastic medium with Poisson's ratio equal to one-fourth and for pressure pulses with a step and exponential distributions.

Plane stress solutions for the transient response of an infinite elastic plate subjected to radial pressure in a circular hole have been given by Kromm [2] and Miklowitz [3]. Kromm's solution involves a numerical evaluation of Volterra's integral equation of the first kind; whereas, Miklowitz developed a special Laplace-transform inversion technique. It is pointed out in [3] that these solutions have application in calculating the tensile circumferential stresses generated by the unloading mechanism of a stretch elastic plate in which a circular hole is suddenly punched. The method of characteristics has also been applied to investigate the propagation of cylindrical waves in plates by Chou and Koenig [4]. All the above methods of solution require lengthy numerical calculations.

In this analysis, the computational difficulties in Selberg's solution are overcome by presenting the location of the poles for all values of Poisson's ratio of elastic media and by noting that the integral along the branch cut can be accurately approximated by an exponential function. Formulae in terms of elementary functions are developed for the circumferential stress, radial displacement, and radial velocity at the cavity wall for a step pressure pulse. These formulae can be easily used as influence coefficients to determine, by means of Duhamel integrals, the response produced by an arbitrary pressure pulse.

FORMAL SOLUTION

An unbounded, elastic medium with a circular, cylindrical cavity is subjected to a circumferentially uniform, step pressure pulse at the cavity surface. The problem is one of axisymmetric plane strain and only waves of dilatation radiate away from the cavity wall. Formal integral solutions for the problem are derived in [5] and are also similar to those derivations presented in [1] and [3]. The formal solutions for the circumferential stress and radial displacement at the cavity boundary are

$$\frac{\sigma_\theta}{p} = \frac{-1}{2\pi i}\int_{-i\infty+b}^{+i\infty+b} \frac{\left[\left(\frac{v}{1-v}\right)sK_0(s) - \left(\frac{1-2v}{1-v}\right)K_1(s)\right] e^{s\tau}\,\mathrm{d}s}{s\left[sK_0(s) + \left(\frac{1-2v}{1-v}\right)K_1(s)\right]}. \tag{1}$$

$$\frac{E(1-v)u}{pa} = \frac{1}{2\pi i}\int_{-i\infty+b}^{+i\infty+b} \frac{K_1(s)\, e^{s\tau}\,\mathrm{d}s}{s\left[K_1(s) + \left(\frac{1-v}{1-2v}\right)sK_0(s)\right]} \tag{2}$$

in which

$$\tau = c_1 t/a \tag{3}$$

where a is the cavity radius, b is a small positive number, c_1 is the velocity of propagation for dilatational waves, E is Young's modulus, K_n are nth ordered Bessel functions of the second kind, p is the magnitude of the step pressure pulse, s is the Laplace transform variable, and v is Poisson's ratio.

RESPONSE FORMULAE FOR PLANE STRAIN

The formal solutions given by equations (1) and (2) are evaluated by considering the contour shown in Fig. 1. The modified Bessel functions require a branch cut along the negative real axis, and Selberg [1] has proved that there are two poles in the complex plane.

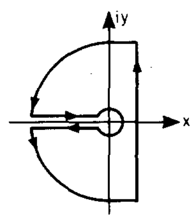

FIG. 1. Path of integration of the s-plane with two enclosed poles.

Then the integrals in the formal solution can be replaced by an integral around the branch cut plus $2\pi i$ times the residues of the poles. The response formulae consist of the long time or static solution obtained from the integration around the origin, a damped oscillatory part from the residues of the poles, and a branch integral along the negative real axis. These branch integrals are recorded in [5] and were evaluated for four values of Poisson's ratio. In each case, it was found that these branch integrals could be accurately approximated by an exponential function. The location of the poles $s_{1,2} = x \pm iy$ are recorded in Fig. 2, and accurate formulae for the circumferential stress, radial displacement, and the particle velocity at the cavity wall can be written in terms of elementary functions in the following

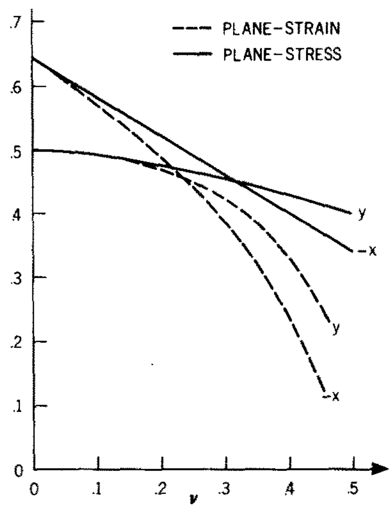

FIG. 2. Location of the poles.

form:

$$\frac{\sigma_\theta}{p} = 1-\frac{2(1-2\nu)\,e^{x\tau}}{A^2+B^2}(A\cos y\tau+B\sin y\tau)+C\,e^{x\tau} \tag{4}$$

$$\frac{E(1-\nu)}{pa}u = 1-\frac{2(1-\nu)(1-2\nu)\,e^{x\tau}}{A^2+B^2}(A\cos y\tau+B\sin y\tau)+D\,e^{x\tau} \tag{5}$$

$$\frac{E}{pc_1}\frac{\partial u}{\partial t} = \frac{-2(1-2\nu)\,e^{x\tau}}{A^2+B^2}[(xA+yB)\cos y\tau+(xB-yA)\sin y\tau]+F\,e^{x\tau} \tag{6}$$

where

$$A = (1-\nu)^2(x^2-y^2)+1-2\nu, \qquad B = 2(1-\nu)^2xy \tag{7}$$

$$C = \frac{2(1-2\nu)A}{A^2+B^2}-\frac{1}{1-\nu} \tag{8}$$

$$D = \frac{2(1-\nu)(1-2\nu)A}{A^2+B^2}-1 \tag{9}$$

$$F = \frac{2(1-2\nu)(xA+yB)}{A^2+B^2}+\frac{1-2\nu}{(1-\nu)^2}\,. \tag{10}$$

RESPONSE FORMULAE FOR PLANE STRESS

Formulae similar to equations (4), (5), and (6) can be obtained for a state of plane stress by modifying the elastic constants; e.g., see [6]. The plane stress solution is obtained by replacing $\nu/(1-\nu)$ by ν, $(1-2\nu)/(1-\nu)$ by $1-\nu$, and c_1 by c, where c is the propagation velocity in a thin plate given by $c = [E/\rho(1-\nu^2)]^{\frac{1}{2}}$. For a state of plane stress the response formulae at the circular hole are

$$\frac{\sigma_\theta}{p} = 1 - \frac{2(1-\nu^2)\,e^{x\tau}}{A^2+B^2}(A\cos y\tau + B\sin y\tau) + C\,e^{x\tau} \tag{11}$$

$$\frac{Eu}{pa(1+\nu)} = 1 - \frac{2(1-\nu)\,e^{x\tau}}{A^2+B^2}(A\cos y\tau + B\sin y\tau) + D\,e^{x\tau} \tag{12}$$

$$\frac{E}{pc}\frac{\partial u}{\partial t} = \frac{-2(1-\nu^2)\,e^{x\tau}}{A^2+B^2}[(xA+yB)\cos y\tau + (xB-yA)\sin y\tau] + F\,e^{x\tau} \tag{13}$$

where

$$A = x^2 - y^2 + 1 - \nu^2, \qquad B = 2xy \tag{14}$$

$$C = \frac{2(1-\nu^2)A}{A^2+B^2} - (1+\nu) \tag{15}$$

$$D = \frac{2(1-\nu)A}{A^2+B^2} - 1 \tag{16}$$

$$F = (1-\nu^2) + \frac{2(1-\nu^2)(xA+yB)}{A^2+B^2}. \tag{17}$$

DISCUSSION AND NUMERICAL RESULTS

Formulae for the circumferential stress, radial displacement, and radial particle velocity at the boundary of a circular, cylindrical cavity or circular hole for the corresponding plane stress problem which is subjected to a step pressure pulse were developed. These formulae contain elementary functions and depend only on the location of the poles in the formal solution and the elastic constants of the medium. The location of the poles is presented as a function of Poisson's ratio in Fig. 2.

Response curves for the circumferential stress, displacement, and velocity at the cavity boundary for Poisson's ratio $\nu = 0, 0{\cdot}25, 0{\cdot}40, 0{\cdot}45$ are presented in [5]. For a state of plane

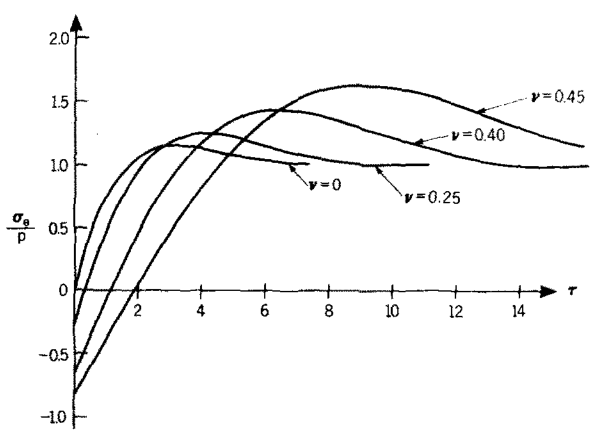

FIG. 3. Circumferential stress at cavity boundary.

stress the same quantities at the boundary of the hole are given for $\nu = 0$, 0·30, 0·50. As previously mentioned, the approximation of the branch integrals by exponential functions is extremely accurate. In fact, no differences beween the exact solution and the formulae given by equations (4), (5), (6) and (11), (12), (13) could be observed in the plotted results. For brevity, only the circumferential stress at the cavity wall are presented; this data is presented in Fig. 3. Response data for other pressure pulses can be calculated by using equations (4), (5), (6) and (11), (12), (13) as influence coefficients.

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Абстракт—Весконечное упругое тело с круглой цилиндрической полостью, находится под влиянием внезапно приложенного, постоянного на окружности, импульсного давления на поверхности полости. Выводятся формулы для окружного напряжения, радиального перемещения и радиальной скорости на поверхности полости. Эти формулы содержат только элементарные функции, и могут быть легко использованы, как влияние коэффициентов, для определения, в смысле интегралов Дюамеля, реакции, созданной произвольным импульсом давления.